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MOUNTAIN IMPASSE THEOREM AND SPECTRUM OF SEMILINEAR ELLIPTIC PROBLEMS

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ABSTRACT. This paper studies a minimax problem for functionals in Hilbert space in the form of $G(u) = \frac{1}{2}\rho\|u\|^2 - g(u)$, where $g(u)$ is Fréchet differentiable with weakly continuous derivative. If G has a "mountain pass geometry" it does not necessarily have a critical point. Such a case is called, in this paper, a "mountain impasse". This paper states that in a case of mountain impasse, there exists a sequence $u_j \in H$ such that

$$g'(u_j) = \rho_j u_j, \qquad \rho_j \to \rho, \ \|u_j\| \to \infty,$$

and $G(u_j)$ approximates the minimax value from above. If

$$\gamma(t) = \sup_{\|u\|^2=t} g(u)$$

and

$$J_0 = \left(2 \inf_{t_2>t_1>0} \frac{\gamma(t_2)-\gamma(t_1)}{t_2-t_1}, 2 \sup_{t_2>t_1>0} \frac{\gamma(t_2)-\gamma(t_1)}{t_2-t_1}\right),$$

then $g'(u) = \rho u$ has a nonzero solution u for a dense subset of $\rho \in J_0$.

1. FORMULATION OF RESULTS

If g is a C^1-functional on a Hilbert space and u is a critical point of g on a sphere, then $g'(u) = \rho u$, $\rho \in \mathbf{R}$. This approach to semilinear elliptic equations has been known for decades (cf. [1, 2]), but a question of the range of ρ has remained open. A recent series of papers (cf. [7] and references therein) provides an answer that can be summarized as follows.

Let H be an infinite dimensional Hilbert space and let $g: H \to \mathbf{R}$ be a C^1-map (with respect to Fréchet differentiation). Let H_w be the space H supplied with the weak topology. Assume that

$$g \in C(H_w \to \mathbf{R}), \tag{1.1}$$

$$g' \in C(H_w \to H). \tag{1.2}$$

(By continuity we always mean local contiuity without uniform bounds.)

In applications to semilinear elliptic problems condition (1.1)–(1.2) correspond to the subcritical growth of the right-hand side.

Consider the following function

$$\gamma(t) = \sup_{\|u\|^2=t} g(u). \tag{1.3}$$

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Theorem 1.1. *Assume* (1.1), (1.2). *The function* (1.3) *is continuous, nondecreasing and possesses left- and right-hand derivatives* $\gamma'_-(t) \le \gamma'_+(t)$. *For every* $t > 0$ *such that* $\gamma'_+(t) \neq 0$ $(\gamma'_-(t) \neq 0)$ *there exist* $u_+ \in H$ $(u_- \in H)$ *such that*

$$\|u_\pm\|^2 = t, \qquad g(u_\pm) = \gamma(t), \tag{1.4}$$

and

$$2\gamma'_\pm(t) u_\pm = g'(u_\pm). \tag{1.5}$$

In other words, the spectrum $\{\rho\}$ of the problem

$$\rho u = g'(u), \qquad u \in H\backslash\{0\}, \tag{1.6}$$

contains all the tangent slopes from the graph of $2\gamma(t)$. If $\gamma(t)$ had a continuous derivative, then (1.6) would be solvable for all

$$\rho \in J_0 = \left(\inf_{t>0} 2\gamma'_-(t), \sup_{t>0} 2\gamma'_+(t)\right). \tag{1.7}$$

However, $\gamma(t)$ is not necessarily differentiable (cf. [7]). On the other hand, [7] gives sufficient conditions for (1.6) to be solvable with $\rho \in J_0$. The argument of [7] involves a mountain pass lemma with an additional condition of the (PS)-type which we avoid here.

The main result of this paper is

Theorem 1.2. *Let* g *satisfy* (1.1), (1.2). *Then for every* $\rho \in J_0$ *such that* (1.6) *is not solvable there is a sequence* $\rho_j \in J_0$ *and a sequence* $u_j \in H$, *such that*

$$\rho_i > \rho, \qquad \rho_j \to \rho, \tag{1.8}$$

$$\rho_j u_j = g'(u_j), \qquad \|u_j\| \to \infty. \tag{1.9}$$

Theorem 1.2 means that the set of eigenvalues of (1.6) is dense in J_0 and that a missing eigenvalue always can be approximated by a "blow up" sequence. Theorem 1.2 reflects a technical situation that might be called a *mountain impasse.* This is the case when a functional has a standard mountain pass geometry but no critical points. This situation is handled by Theorem 2.1 which is a refinement of Schechter's Mountain Pass Alternative [4]. Section 2 contains the proof of Theorem 2.1. Section 3 discusses applications to semilinear elliptic problems. The tangible benefit of Theorem 1.2 is not a mountain impasse itself (which to our best understanding was never observed in elliptic problems), but a relation between solvability and priori bounds, widely used before in the topological approach. We will discuss this in more detail at the end of §3.

2. Mountain impasse theorem

Let

$$G \in C^1(H \to \mathbf{R}), \qquad G' \in C(H_w \to H_w) \tag{2.1}$$

be a weakly lower semicontinuous functional with a mountain pass geometry, as follows. Let $\delta > 0$, $t_0 > 0$, $e \in H$, $\|e\|^2 > t_0$ and assume that

$$G(u) \ge 2\delta > 0 \text{ for } \|u\|^2 = t_0, \text{ while } G(0), G(e) \le 0. \tag{2.2}$$

We assume that G has no critical points:

$$G'(u) \neq 0 \quad \text{when } G(u) \ge \delta. \tag{2.3}$$

The following condition will also be required:

$$\text{if } u_k \xrightarrow{w} u_0, \ \limsup (G'(u_k), u_k) \le 0 \text{ and } G'(u_k) - (G'(u_k), u_k)u_k/\|u_k\|^2 \to 0, \text{ then } u_k \to u_0 \text{ in } H. \tag{2.4}$$

We should note that this condition of a weak (PS) type becomes a mere weak continuity condition when G is as in (3.2). Let

$$S_t = \{u \in H : \|u\|^2 = t\}, \qquad B_t = \{u \in H : \|u\|^2 \le t\}. \tag{2.5}$$

For every $t > \|e\|^2$ we define $\Phi(t)$ as a collection of paths $\varphi \in C([0, 1] \to B_t)$ such that

$$\varphi(0) = 0, \qquad \varphi(1) = \varepsilon. \tag{2.6}$$

Let

$$\kappa(t) = \inf_{\varphi \in \Phi(t)} \max_{s \in [0, 1]} G(\varphi(s)). \tag{2.7}$$

From (2.2) it follows that

$$\kappa(t) \ge 2\delta \quad \text{when } t > \|e\|^2. \tag{2.8}$$

Theorem 2.1. *Assume* (2.1)–(2.4). *There exist a sequence* $\alpha_j > 0$, $\alpha_j \to 0$, *and a sequence* $u_j \in H\backslash\{0\}$, $\|u_j\| \to \infty$, *such that*

$$G'(u_j) = -\alpha_j u_j, \tag{2.9}$$

$$G(u_j) \ge \delta. \tag{2.10}$$

The proof of Theorem 2.1 will be given as a sequence of lemmas. Relations (2.1)–(2.4) are assumed throughout §2. The following statement can be found in [5].

Lemma 2.2. *Let* $Z(u) \in C(B_t \to H)$ *and* $Z(u) \ne 0$ *on* $B_t\backslash\{0\}$. *Assume that there is a closed subset* Q *of* $B_t\backslash\{0\}$ *and a* $\theta < 1$ *such that*

$$(Z(u), u) + \theta\|Z(u)\|\,\|u\| \ge 0, \qquad u \in Q. \tag{2.11}$$

Then for each $\alpha < (1 - \theta)$ *there is a locally Lipschitz mapping* $Y(u) : B_t\backslash\{0\} \to H$ *such that*

$$(Z(u), Y(u)) \ge \alpha\|Z(u)\|, \qquad u \in B_t\backslash\{0\}, \tag{2.12}$$

$$(Y(u), u) > 0, \qquad u \in Q, \tag{2.13}$$

and

$$\|Y(u)\| \le 1, \qquad u \in B_t\backslash\{0\}. \tag{2.14}$$

Lemma 2.3. *Assume that there is an* $\varepsilon > 0$, *such that*

$$G'(u) = \beta u \tag{2.15}$$

has no solution u *when*

$$u \in H_\delta = \{u \in H : G(u) \ge \delta\}, \tag{2.16}$$

and $\beta \in [-2\varepsilon, 0]$. *Then for any* $t > \|e\|^2$ *there exist a* $\theta < 1$ *such that*

$$(G'(u), u) + \theta\|G'(u)\|\,\|u\| \le 0, \qquad u \in H_\delta \cap B_t \Rightarrow (G'(u), u) \le -\varepsilon\|u\|^2. \tag{2.17}$$

Proof. Assume the opposite, namely that there is a sequence $u_j \in H_\delta \cap B_t$ and a sequence $\theta_j \to 1$, $\theta_j < 1$, such that

$$(G'(u_j), u_j) + \theta_j \|G'(u_j)\| \, \|u_j\| \le 0, \tag{2.18}$$

$$\beta(u_j) := (G'(u_j), u_j)/\|u_j\|^2 \in [-\varepsilon, 0]. \tag{2.19}$$

Let u_i be a renamed weakly convergent subsequence, and $u_0 = w\text{-}\lim u_j$. By (2.1) $G'(u_j) \overset{w}{\to} G'(u_0)$ and therefore $G'(u_j)$ is bounded in norm. Then (2.18) easily implies

$$G'(u_j) - \beta(u_j)u_j \to 0. \tag{2.20}$$

Then by (2.4) $u_j \to u_0$ in H, $G(u_0) \ge \delta$, and

$$G'(u_0) = \beta(u_0)u_0, \qquad u_0 \in H_\delta \cap B_t. \tag{2.21}$$

By (2.19) $\beta(u_0) \in [-\varepsilon, 0]$ which contradicts assumptions of the lemma. □

Lemma 2.4. *There is a number* $r(t) > 0$ *and a number* $\mu > 0$ *independent of* t *such that*

$$\|G'(u)\| \ge r(t), \qquad u \in H_\delta \cap B_t, \tag{2.22}$$

$$\|u\| \ge 2\mu, \qquad u \in H_\delta \cap B_t. \tag{2.23}$$

Proof. (1) Assume that (2.22) fails. Then there is a sequence $u_j \overset{w}{\to} u_0 \in B_t$, such that $G'(u_j) \to 0$. Then by (2.4) $u_j \to u_0$ in H, $u_0 \in H_\delta \cap B_t$, and $G'(u_0) = 0$, which contradicts (2.2).

(2) Consider the lower bound of $\|u\|$ on H_δ. If $u_j \to 0$ on H_δ, then $G(u_j) \to G(0) \le 0$ by (2.2), which contradicts the assumption on u_j. □

Lemma 2.5. *Under assumptions of Lemma* 2.3,

$$D_t^+\kappa(t) \le -\tfrac{1}{2}\varepsilon\mu^2/t, \qquad t > \|e\|^2, \tag{2.24}$$

with μ *as in* (2.23).

Proof. (1) Let us define the following sets:

$$\begin{aligned}
Q_0 &= \{u \in B_{2t} : |G(u) - \kappa(t)| \le \delta/2\}, \\
\widetilde{Q}_0 &= \{u \in B_{2t} : |G(u) - \kappa(t)| \ge \delta\}, \\
Q_1 &= \left\{u \in B_{2t} : \frac{(G'(u), u)}{\|G'(u)\| \, \|u\|} \le -1 + \eta\right\}, \\
Q_2 &= \left\{u \in B_{2t} : \frac{(G'(u)u)}{\|G'(u)\| \, \|u\|} \ge -1 + 2\eta\right\},
\end{aligned} \tag{2.25}$$

where $t > \|e\|^2$, $0 < 2\eta < 1 - \theta$ with θ as in Lemma 2.3 applied to the ball B_{2t}. Let

$$\begin{aligned}
\chi_0(u) &= d(u, \widetilde{Q}_0)/(d(u, Q_0) + d(u, \widetilde{Q}_0)), \\
\chi_1(u) &= d(u, Q_2)/(d(u, Q_1) + d(u, Q_2)), \\
\chi_2(u) &= 1 - \chi_1(u), \qquad u \in H.
\end{aligned} \tag{2.26}$$

The functions (2.26) are Lipschitz continuous, their range is $[0, 1]$, they equal one on Q_0, Q_1, Q_2, respectively, and vanish, respectively, on $\widetilde{Q}_0$, Q_2 and Q_1.

We now wish to apply Lemma 2.2 for B_{2t} with $Z = G'$, $Q = \operatorname{supp}\chi_1 \cap \operatorname{supp}\chi_0 = B_{2t}\backslash\{\widetilde{Q}_0 \cup Q_2\}$ and θ as in Lemma 2.3. Consider the initial value problem

$$d\sigma/dh = \chi_0(\sigma)\chi_1(\sigma)\sigma - N\chi_0(\sigma)\chi_2(\sigma)Y(\sigma)/\|Y(\sigma)\|, \tag{2.27}$$

$$\sigma(h)|_{h=0} = \varphi, \qquad \varphi \in H, \qquad N = 2\varepsilon t/(1-2\eta)r. \tag{2.28}$$

The right-hand side in (2.27) is locally Lipschitz continuous in σ and thus the problem has a unique C^1-solution σ defined for all h. Note that if $\varphi = 0$ or $\varphi = e$, then so is σ for all h.

(2) Let $\varphi_j \in \Phi(t)$ be a minimizing sequence for (2.7) and σ_j be correspondent solutions of (2.27)–(2.28). Then for $t_1 \in (t, 2t)$,

$$\kappa(t_1) \le \sup_{s\in[0,1]} G(\sigma_j(h; s)) \tag{2.29}$$

as long as

$$\|\sigma_j(h; s)\|^2 < t_1 \quad \text{for all } s \in [0, 1]. \tag{2.30}$$

Let us establish a bound on h that implies (2.30). By (2.27)

$$\frac{d}{dh}\|\sigma_j\|^2 = 2\chi_0\chi_1\|\sigma_j\|^2 - 2\chi_0\chi_2 N(Y(\sigma_j), \sigma_j)\|Y(\sigma_j)\|^{-1} \le 2\|\sigma_j\|^2. \tag{2.31}$$

Therefore,

$$\|\sigma_j(h; s)\|^2 \le te^{2h}. \tag{2.32}$$

Consequently, assuming

$$0 \le h \le \tfrac{1}{2}\ln(t_1/t) \tag{2.33}$$

we obtain (2.30). In the further course of the proof h will be subject to additional bounds from above.

(3) Let us estimate the derivative of $G(\sigma)$.

$$\frac{d}{dh}G(\sigma_j) = \chi_0\chi_1(G'(\sigma_j), \sigma_j) - N\chi_0\chi_2(G'(\sigma_j), Y(\sigma_j))/\|Y(\sigma_j)\|. \tag{2.34}$$

By Lemma 2.2 as already applied,

$$(G_j'(\sigma), Y(\sigma_j))/\|Y(\sigma_j)\| \ge (1-2\eta)\|G'(\sigma_j)\| \quad \text{when } \sigma \in \operatorname{supp}\chi_0\chi_2. \tag{2.35}$$

By Lemma 2.3

$$(G'(\sigma_j), \sigma_j) \le -\varepsilon\|\sigma_j\|^2 \quad \text{when } \sigma \in \operatorname{supp}\chi_0\chi_1. \tag{2.36}$$

Then (2.34) yields

$$\begin{aligned} \frac{d}{dh}G(\sigma_j) &\le -\varepsilon\chi_0\chi_1\|\sigma_j\|^2 - (1-2\eta)N\chi_0\chi_2\|G'(\sigma_j)\| \\ &\le -\varepsilon\chi_0\chi_1\|\sigma_j\|^2 - (1-2\eta)N\chi_0\chi_2 r\|\sigma_j\|^2/2t \\ &\le -\varepsilon\chi_0(\chi_1+\chi_2)\|\sigma_j\|^2 = -\varepsilon\chi_0(\sigma_j)\|\sigma_j\|^2. \end{aligned} \tag{2.37}$$

(4) Consider the following sets of $s \in [0, 1]$. Let

$$I_1 = \{s \in [0, 1] : |G(\varphi(s)) - \kappa(t)| \ge \delta/2\}. \tag{2.38}$$

For j large enough the inequality in (2.38) holds only if $G(\varphi_j(s)) \le \kappa(t) - \delta/2$, since φ_j is a minimizing sequence and $\kappa(t)$ is approximated by the maximal values of $G(\varphi_j(s))$. By (2.37), $G(\sigma_j(h; s)) \le \kappa(t) - \delta/2$ for $s \in I_1$. Now let I_2 be a subset of $[0, 1]\backslash I_1$, such that $\sigma_j(h; s) \in Q_0$ for all $h \in [0, h_1]$, $h_1 := \frac{1}{2}\ln(t_1/t)$, and $I_3 = [0, 1]\backslash(I_1 \cup I_2)$. On I_2 (2.37) implies

$$\frac{d}{dh}G(\sigma_j) \le -\varepsilon\|\sigma_j\|^2. \tag{2.39}$$

By (2.31)

$$\frac{d}{dh}\|\sigma_j\|^2 \ge -2N\|\sigma_j\| \tag{2.40}$$

and consequently,

$$\|\sigma_j\| \ge \|\varphi_j\| - Nh, \qquad s \in I_2. \tag{2.41}$$

Since $\varphi_j(s) \in H_\delta$ when $s \in I_2$, Lemma 2.4 implies

$$\|\sigma_j\| \ge 2\mu - Nh \tag{2.42}$$

and assuming

$$h \le \mu/N, \tag{2.43}$$

one has

$$\|\sigma_j\| \ge \mu \quad \text{for } s \in I_2, \tag{2.44}$$

and therefore,

$$\frac{d}{dh}G(\sigma_j) \le -\varepsilon\mu^2, \qquad s \in I_2. \tag{2.45}$$

Finally, if $s \in I_3$, let $h_0 \in [0, h_1]$ be a maximal h, such that $\sigma_j(h; s) \in Q_0$ for $h \in [0, h_0]$. Then

$$G(\sigma_j(h; s)) \le G(\sigma_j(h_0, s)) = \kappa(t) - \delta/2, \qquad s \in I_3. \tag{2.46}$$

Combining (2.38), (2.45), and (2.44), one has from (2.29) that

$$\begin{aligned} \kappa(t_1) &\le \max_{s\in[0,1]} G(\sigma_j(h; s)) \\ &\le \max\left\{\kappa(t) - \delta/2, \max_{s\in I_3} G(\varphi_j) - \varepsilon\mu^2 h\right\} \\ &\le \max\{\kappa(t) - \delta/2, m_j - \varepsilon\mu^2 h\}, \end{aligned} \tag{2.47}$$

where $m_j = \max_{s\in[0,1]} G(\varphi_j)$, $m_j \to \kappa(t)$. With j large enough and an additional upper bound on h

$$h < \delta/6\varepsilon\mu^2, \tag{2.48}$$

one has

$$\kappa(t_1) \le \kappa(t) - \varepsilon\mu^2 h. \tag{2.49}$$

Relation (2.49) is valid only as far as h satisfies restrictions (2.48), (2.43), and (2.33). Three of them can be reduced to (2.33) when

$$t_1 < \min\{2t, te^{2h_2}, te^{2h_3}\}, \quad \text{where } h_2 = \frac{\mu}{N} \text{ and } h_3 = \frac{\delta}{6\varepsilon\mu^2}. \tag{2.50}$$

Then (2.49) with $h = \frac{1}{2}\ln(t_1/t)$ immediately implies (2.24). □

Proof of Theorem 2.1. It is already proved in [4] that for any $t > \|\varepsilon\|^2$ there is a $u \in H_\delta \cap B_t$ and $\alpha > 0$ such that

$$G'(u) = -\alpha u. \tag{2.51}$$

Assume that there are no sequences $\alpha_j > 0$, $u_j \in H_\delta$ satisfying (2.51) and such that $\alpha_j \to 0$. Then the conditions of Lemma 2.3 are satisfied with some $\varepsilon > 0$ and by Lemma 2.5

$$\limsup_{t\to\infty} \kappa(t) \to -\infty, \tag{2.52}$$

which contradicts (2.8). Therefore $\{\alpha_j\}$ necessarily has a subsequence with zero limit. Assume now that $\{u_j\}$ has a bounded subsequence. Then there is a weakly convergent renamed subsequence $u_j \overset{w}{\to} u_0$ and

$$G'(u_j) \to 0. \tag{2.53}$$

Then by (2.4) $u_j \to u_0$, $G(u_0) \ge \delta$, and $G'(u_0) = 0$, which contradicts (2.3). Thus $\|u_j\| \to \infty$. □

3. Applications to elliptic problems

We wish to show now that Theorem 2.1 implies Theorem 1.2. Our argument is somewhat repetitious of [7] and we omit details.

(1) Let

$$\rho_0 \in J_0, \tag{3.1}$$

and

$$G(u) = \tfrac{1}{2}\rho_0\|u\|^2 - g(u) + c, \qquad c \le g(0). \tag{3.2}$$

It is easy to see that if g satisfies (1.1) and (1.2), then G satisfies (2.1).

(2) Let us verify (2.2). Let

$$\Gamma(t) = \tfrac{1}{2}\rho_0 t - \gamma(t) + c. \tag{3.3}$$

If the function $\Gamma(t)$ has a point of a local minimum on $(0, \infty)$, then G has a nonzero critical point (cf. [6] or [7]), which contradicts the assumptions. If the function $\Gamma(t)$ is monotone, then by Theorem 1.2 all the derivatives of $\gamma(t)$ must be either greater or smaller than $\frac{1}{2}\rho_0$ which contradicts (3.1).

The remaining possibility is that $\Gamma(t)$ has a global maximum at $t = t_0 \in (0, \infty)$. Then (2.2) is satisfied with the following choices. Let $t_1 > t_0$ and let e be an element of a maximizing sequence for g on S_{t_1}, such that

$$\tfrac{1}{2}\rho_0 t_1 - g(e) + c < M + c, \tag{3.4}$$

where

$$M = \tfrac{1}{2}\rho_0 t_0 - \gamma(t_0). \tag{3.5}$$

Finally, set

$$\delta = \tfrac{1}{2}\min(M - \tfrac{1}{2}\rho_0\|e\|^2 + g(e), M + g(0)) \tag{3.6}$$

and

$$c = 2\delta - M. \tag{3.7}$$

If one assumes that (1.6) has no solution with $\rho = \rho_0$, then G satisfies (2.3). Now note that (2.4) follows from (1.2) and one can apply Theorem 2.1. □

We discuss here only the applications to the problem

$$-\rho\Delta u = f(u), \qquad u \in H_0^1(\Omega)\backslash\{0\}, \tag{3.8}$$

where $\Omega \subset \mathbf{R}^n$, $n \geq 3$, is an open bounded set, $f : \mathbf{R} \to \mathbf{R}$ is continuous and subcritical at infinity, i.e.,

$$f(s) = o(|s|^{(n+2)/(n-2)}) \quad \text{as } s \to \infty. \tag{3.9}$$

Let

$$F(s) = \int_0^s f(\sigma)\,d\sigma, \tag{3.9}$$

$$g(u) = \int_\Omega F(u)\,dx. \tag{3.10}$$

It is well known that g satisfies (1.1) and (1.2) on $H_0^1(\Omega)$. Equation (3.8) is an equation for a critical point of g on

$$S_t = \left\{u \in H_0^1(\Omega) : \int_\Omega |\nabla u|^2 = t\right\}. \tag{3.11}$$

Theorem 1.2 then implies that (3.8) is solvable for a dense subset of $\rho \in J_0$. The interval J_0 is defined here as an open interval between the lower and upper bound of the slopes on the graph of

$$2\gamma(t) = \sup_{u\in S_t} 2\int_\Omega F(u)\,dx. \tag{3.12}$$

One can reverse Theorem 1.2 and state the solvability of (1.6) for all $\rho \in J_0$ with an additional condition of an a priori bound on a Hilbert norm of u.

Theorem 3.1. *Assume* (1.1), (1.2), *and* (1.8). *If there is a* $\nu > 0$ *and a* $c > 0$, *such that for any* u *satisfying* (1.6) *with* $|\rho - \rho_0| < \nu$,

$$\|u\| \leq c, \tag{3.13}$$

then (1.6) *has a solution with* $\rho = \rho_0$.

This statement is an elementary corollary of Theorem 1.2.

There are several important results which establish a priori bounds in L^∞ motivated by the topological approach to (3.8) (cf. [3] and references therein). The following statement uses an argument from [3].

Corollary 3.2. *Let* $n \geq 3$, Ω *be starshaped,* $f \geq 0$, *and let*

$$\frac{F(s)}{s^\sigma} \text{ be a decreasing function near } s = +\infty \text{ with some } \sigma < \frac{2n}{n-2}. \tag{3.14}$$

Then the problem (3.8) *satisfies the condition* (3.14) *and consequently is solvable for all* $\rho \in J_0$.

Proof. Since Ω is starshaped, a well-known Pohozhaev-Rellich identity (resulting from multiplication of (3.8) by $(x \cdot \nabla)u$) provides

$$\rho\|u\|^2 \leq \frac{2n}{n-2}\int F(u). \tag{3.15}$$

At the same time, multiplication of (3.8) by u gives

$$\rho\|u\|^2 = \int f(u)u. \tag{3.16}$$

From (3.15) and since the problem might have only positive solutions, one has

$$\int F(u) \leq \sigma \int f(u)u + c, \qquad c \in \mathbf{R}. \tag{3.17}$$

Then (3.16)–(3.18) immediately provide (3.14). □

This corollary includes cases of f with sub- (or super-)linear behavior both at 0 and at ∞.

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